

On the dynamics of a particle on a cone

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Abstract

A detailed study of the classical and quantum mechanics of a free particle on a double cone and the particle bounded to its tip by the harmonic oscillator potential is presented.

Keywords: Quantum mechanics; Classical mechanics; Constrained systems; Motion on a cone; Harmonic oscillator; Classical and quantum unstable motion.

1. Introduction

The quantum mechanics on conical spaces has attracted much attention not only in early investigations [1, 2] but also in recent papers [3, 4, 5, 6, 7]. Among different motivations for the study of a quantum particle on a cone we only recall the context of the $2 + 1$ dimensional quantum gravity [1], cosmic strings [8] and defects in various media [9]. Nevertheless, to our best knowledge, all investigations of quantum mechanics on a cone were restricted to the case of the cone with a single nappe or equivalently the plane with a deficit angle and the situation of a double cone, i.e. two cones placed apex to apex, has not yet been pursued. This is all the more surprising since the case of a double cone is much more simple. In particular, in contrast to the situation with a single nappe, we have no troublesome questions like that concerning behavior of a free particle moving in a generator towards a cone tip after reaching it, and there is no need to analyze self-adjoint extensions of symmetric operators representing observables of the system. In this work we perform a detailed analysis of the classical and quantum mechanics on a double cone involving the case of a free particle and harmonic oscillator on the cone. An interesting feature of the dynamics on the double cone discussed in this paper is the instability of the classical rectilinear motion on

the meridian in the sense of Lyapunov and its implications on the quantum level. The paper is organized as follows. In Sec. II we study the dynamics of the free classical particle on the cone. Section III is devoted to the quantum mechanics on the cone in the case of the free motion. The classical harmonic oscillator on the cone is discussed in Sec. IV. In Sec. V we investigate the quantization of the harmonic oscillator on the cone.

2. Classical mechanics of a free particle on a cone

Let us consider a particle confined to a surface of a circular double cone given by

$$\begin{aligned}x_1 &= l \sin \alpha \cos \varphi, \\x_2 &= l \sin \alpha \sin \varphi, \\x_3 &= l \cos \alpha,\end{aligned}\tag{2.1}$$

where $l \in (-\infty, \infty)$ is the coordinate of a particle on a meridian (generator), 2α is the opening angle of the cone, so $\alpha \in (0, \frac{\pi}{2})$, and $\varphi \in [0, 2\pi)$ specifies the position of a particle on a parallel. In view of (2.1) the equation of the conical surface is of the form

$$x_1^2 + x_2^2 - \tan^2 \alpha x_3^2 = 0.\tag{2.2}$$

The Lagrangian of the free particle with mass m constrained to the conical surface (2.2) is

$$L = \frac{m\dot{l}^2}{2} + \frac{m \sin^2 \alpha l^2 \dot{\varphi}^2}{2}.\tag{2.3}$$

The corresponding Hamiltonian can be written as

$$H = \frac{p_l^2}{2m} + \frac{p_\varphi^2}{2ml^2 \sin^2 \alpha},\tag{2.4}$$

where $p_l = \frac{\partial L}{\partial \dot{l}} = m\dot{l}$ is the momentum referring to the motion in the meridian and $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = ml^2 \sin^2 \alpha \dot{\varphi}$ is the angular momentum. Therefore, the Hamilton's equations are

$$\begin{aligned}\dot{l} &= \frac{p_l}{m}, \\ \dot{\varphi} &= \frac{p_\varphi}{ml^2 \sin^2 \alpha}, \\ \dot{p}_l &= \frac{p_\varphi^2}{ml^3 \sin^2 \alpha}, \\ \dot{p}_\varphi &= 0.\end{aligned}\tag{2.5}$$

Now let $J = p_\varphi = \text{const}$ designates the conserved angular momentum. For $J = 0$ we find from (2.5) $\varphi = \text{const}$, $p_l = \text{const}$, and

$$l = l_0 + \frac{p_l}{m}t. \quad (2.6)$$

Therefore $J = 0$ refers to the rectilinear uniform motion along a meridian.

Consider now the case $J \neq 0$. Taking into account the following expression on the energy

$$E = \frac{m\dot{l}^2}{2} + \frac{J^2}{2ml^2 \sin^2 \alpha}, \quad (2.7)$$

we find

$$l = \pm \sqrt{\frac{2E}{m}(t + C)^2 + \frac{J^2}{2mE \sin^2 \alpha}}, \quad (2.8)$$

where $+$ ($-$) sign refers to the motion on the upper (lower) nappe of the cone and C is an integration constant which can be fixed with the help of the initial data. Thus, it turns out that for $J \neq 0$ the particle resides definitely on the upper or lower nappe and there is no communication between these two regions of the configuration space. This means, among others, that the solution to (2.5) with $p_\varphi = J = 0$ referring to rectilinear motion across the tip of the cone is unstable in the Lyapunov sense. Indeed, an arbitrary small perturbation of the initial condition $J = 0$ leads to large deviations of corresponding solutions. Such instability is illustrated in Fig. 1 (bottom-right figure). On the other hand, an immediate consequence of (2.8) is the following inequality:

$$|l| \geq \frac{|J|}{\sqrt{2mE} \sin \alpha}, \quad (2.9)$$

which means that for $J \neq 0$ the geodesics have the lower (upper) bound on the upper (lower) nappe. We remark that the limit $J \rightarrow 0$ for (2.8) is

$$\lim_{J \rightarrow 0} l(t) = \left| \frac{p_{l0}}{m}t + l_0 \right|, \quad (2.10)$$

where we confined for brevity to the case of the motion on the upper nappe. Clearly this asymptotics does not correspond to any real motion. If so it would violate the uniqueness of the initial conditions problem (Cauchy problem) related to the system (2.5). An example of the free motion on the cone (geodesic) given by (2.1) and (2.5) is presented in Fig 1. Further, from the

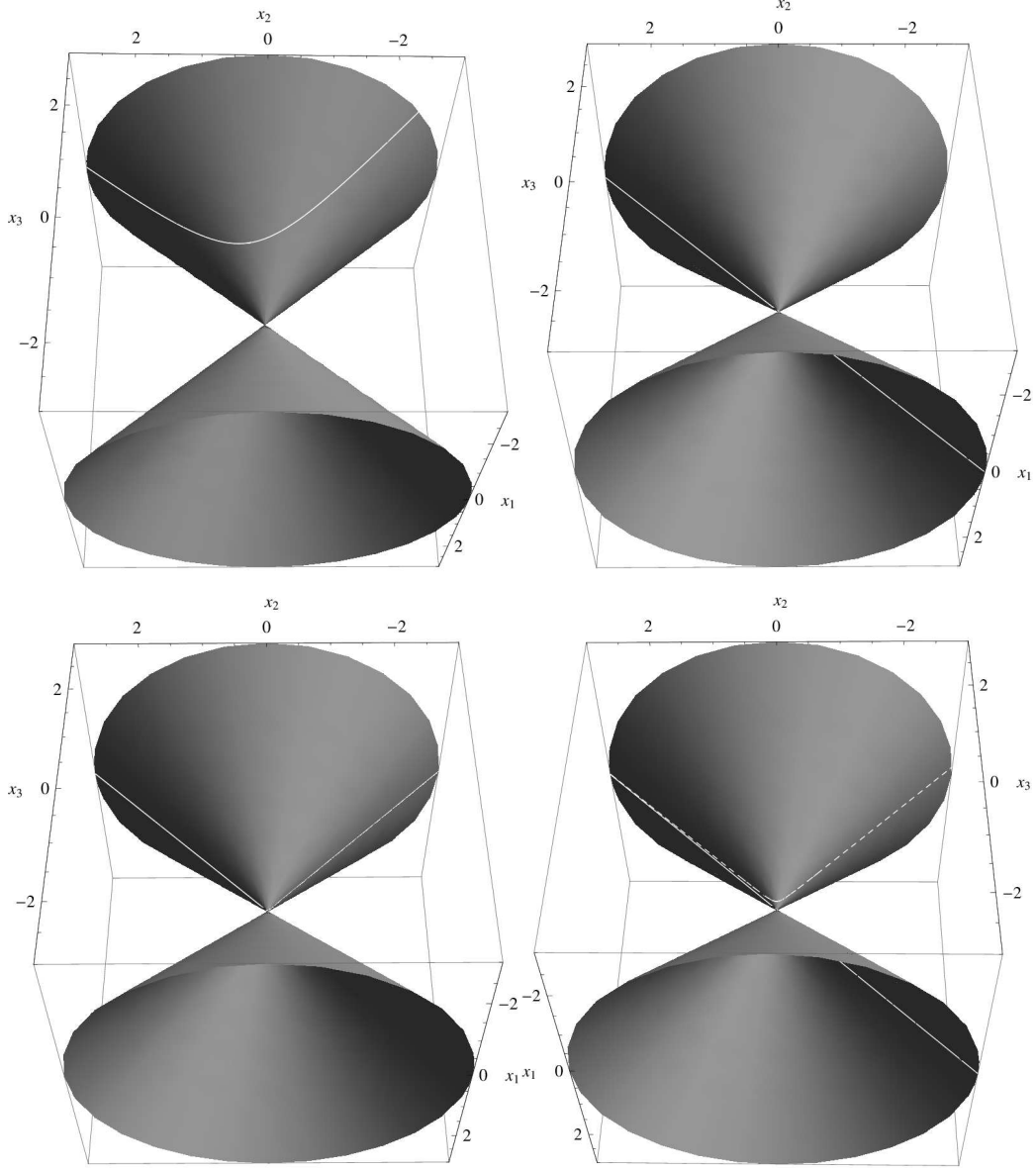


Figure 1: Top left: the trajectory (geodesics) on the cone (2.1) which is the solution of (2.5) with the initial data $l_0 = 5$, $\varphi_0 = \pi/2$, $p_{l0} = -1$, $J = 1$, and the parameters $\alpha = \pi/4$ and $m = 1$. Top right: the rectilinear motion in the meridian of the cone (geodesics) referring to $J = 0$ (see (2.6)). The parameters and remaining initial conditions are the same as in the figure on the left. Bottom left: the trajectory given by (2.1) and (2.5) with the same parameters and initial data as in the figures above besides of $J = 0.01$ corresponding to the limit (2.10) and (2.13). Bottom right: solid line: the solution to (2.5) with $J = 0$ presented in top-right figure. Dashed line: the solution to (2.5) with the same parameters and initial conditions as in the top-left figure besides $J = 0.1$.

second equation of (2.5) we get

$$\varphi - \varphi_0 = 2E \sin \alpha \left[\arctg \frac{2E \sin \alpha}{J} (t + C) - \arctg \frac{2E \sin \alpha}{J} C \right], \quad \text{for } J \neq 0, \quad (2.11)$$

and

$$\varphi = \text{const}, \quad \text{for } J = 0. \quad (2.12)$$

The limit $J \rightarrow 0$ for the angle (2.11) is

$$\varphi - \varphi_0 = 0 \quad \text{or} \quad |\varphi - \varphi_0| = \pi. \quad (2.13)$$

Finally from the first equation of (2.5) and (2.8) we find

$$p_l = \pm \frac{2E(t + C)}{\sqrt{\frac{2E}{m}(t + C)^2 + \frac{J^2}{2mE \sin^2 \alpha}}}, \quad \text{for } J \neq 0. \quad (2.14)$$

By means of (2.8) p_l vanishes when the minimal value is reached by $|l|$ specified by (2.9). The limit $J \rightarrow 0$ for the momentum in the case of the upper nappe can be written as

$$\lim_{J \rightarrow 0} p_l(t) = |p_{l0}| \varepsilon \left(t + \frac{ml_0}{p_{l0}} \right), \quad (2.15)$$

where $l_0 > 0$ and $\varepsilon(x)$ is the sign function.

3. Quantum mechanics of a free particle on a cone

In quantum mechanics the dynamics of a free particle on a double cone defined by the Hamiltonian (2.4), is described by the Schrödinger equation

$$i \frac{\partial f(l, \varphi; t)}{\partial t} = \left(\frac{\hat{p}_l^2}{2m} + \frac{\hat{J}^2}{2m \hat{l}^2 \sin^2 \alpha} \right) f(l, \varphi; t), \quad (3.1)$$

where \hat{p}_l , \hat{l} , and \hat{J} are the momentum, position and angular momentum operators, respectively, and we set $\hbar = 1$. The Hilbert space for a quantum particle on a cone is specified by the scalar product

$$\langle f | g \rangle = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dl |l| f^*(l, \varphi) g(l, \varphi), \quad (3.2)$$

where $|l|dl d\varphi$ coincides up to the multiplicative constant with the surface element of the cone defined by (2.1) such that $dS = \sin \alpha |l| dl d\varphi$. The operators \hat{p}_l , \hat{l} and \hat{J} act in the representation (3.2) in the following way:

$$\begin{aligned}\hat{p}_l f(l, \varphi) &= -i \left(\frac{\partial}{\partial l} + \frac{1}{2l} \right) f(l, \varphi), \\ \hat{l} f(l, \varphi) &= l f(l, \varphi), \\ \hat{J} f(l, \varphi) &= -i \frac{\partial}{\partial \varphi} f(l, \varphi).\end{aligned}\tag{3.3}$$

We remark that the action of the operator \hat{p}_l different from the Schrödinger representation ensures its hermicity with respect to the scalar product (3.2). Of course, such form of the operator \hat{p}_l preserves the canonical commutation relations of \hat{p}_l and \hat{l} as well. The counterpart of the operator \hat{p}_l in the case of the cone with a single nappe was introduced in reference 4. Nevertheless, it is not self-adjoint so it cannot be regarded as a physical observable. We stress that in opposition to the case of the single nappe (or the plane with deficit angle) where the observables such as for example energy are labelled by parameters related to their self-adjoint extensions [4, 5], the operators representing physical observables for the quantum mechanics on the double cone discussed herein, are defined uniquely. We finally point out that the analysis of self-adjoint extensions in the case of the cone with a single nappe is closely related to the study of self-adjoint extensions describing a quantum particle on a plane with extracted point performed by us in reference 10.

Consider now the eigenvalue equation for the Hamiltonian $\hat{H} f(l, \varphi) = E f(l, \varphi)$ following directly from (3.1) and (3.3)

$$\frac{1}{2m} \left(-\frac{\partial^2}{\partial l^2} - \frac{1}{l} \frac{\partial}{\partial l} + \frac{1}{4l^2} - \frac{1}{l^2 \sin^2 \alpha} \frac{\partial^2}{\partial \varphi^2} \right) f_E(l, \varphi) = E f_E(l, \varphi).\tag{3.4}$$

On separating variables we find

$$f_{j,E}(l, \varphi) = e^{ij\varphi} u_{j,E}(l),\tag{3.5}$$

where $f_{j,E}(l, \varphi)$ are the common eigenvectors of the Hamiltonian and the operator of the angular momentum \hat{J} , and $u_{j,E}(l)$ satisfies the equation

$$\frac{d^2 u_{j,E}(l)}{dl^2} + \frac{1}{l} \frac{du_{j,E}(l)}{dl} + \left(2mE - \frac{\frac{1}{4} + \frac{j^2}{\sin^2 \alpha}}{l^2} \right) u_{j,E}(l) = 0.\tag{3.6}$$

The solution to (3.6) for $j \neq 0$ with convergent norm (in a distributive sense) is of the form

$$u_{j,E}(l) = C J_{\sqrt{\frac{1}{4} + \frac{j^2}{\sin^2 \alpha}}}(\sqrt{2mEl}), \quad (3.7)$$

where $J_\nu(z)$ is the Bessel function of the first kind and C is a normalization constant. Now, bearing in mind the behavior of the classical free particle for $J \neq 0$ discussed in previous section, taking into account that $J_\nu(0) = 0$ for $\nu > 0$, and the identity

$$J_\nu(-z) = e^{\nu\pi i} J_\nu(z), \quad (3.8)$$

it seems plausible to identify the case $l > 0$ ($l < 0$) in (3.7) with the quantum particle on the upper (lower) nappe. Let us designate by $u_{+j,E}(l, \varphi)$ ($u_{-j,E}(l, \varphi)$) the corresponding solution to (3.6), so

$$u_{+j,E}(l) = \begin{cases} J_{\sqrt{\frac{1}{4} + \frac{j^2}{\sin^2 \alpha}}}(\sqrt{2mEl}), & \text{for } l > 0, \\ 0, & \text{for } l \leq 0, \end{cases} \quad (3.9)$$

$$u_{-j,E}(l) = \begin{cases} 0, & \text{for } l \geq 0, \\ J_{\sqrt{\frac{1}{4} + \frac{j^2}{\sin^2 \alpha}}}(\sqrt{2mEl}), & \text{for } l < 0. \end{cases} \quad (3.10)$$

In general case the state of a particle on a cone is the superposition of the states $u_{+j,E}$ and $u_{-j,E}$, so we finally arrive at the solution to (3.6) such that

$$f_{j,E}(l, \varphi) = e^{ij\varphi} [A u_{+j,E}(l) + B u_{-j,E}(l)], \quad j \neq 0, \quad (3.11)$$

where A and B are constant. Taking into account the so called “closure equation” [11] of the form

$$\int_0^\infty x J_\nu(\alpha x) J_\nu(\beta x) dx = \frac{1}{\alpha} \delta(\alpha - \beta), \quad \nu > -\frac{1}{2}, \quad (3.12)$$

and (3.8) we find for the eigenvectors of the Hamiltonian (3.11) normalized as

$$\langle f_{j,E} | f_{j',E'} \rangle = \int_0^{2\pi} d\varphi \int_{-\infty}^\infty dl |l| f_{j,E}^*(l, \varphi) f_{j',E'}(l, \varphi) = \delta_{jj'} \delta(\sqrt{E} - \sqrt{E'}), \quad (3.13)$$

the relation

$$|A|^2 + |B|^2 = \frac{m\sqrt{E}}{\pi}. \quad (3.14)$$

Now, it can be easily checked that for $j = 0$ the eigenvectors of the Hamiltonian satisfying (3.6) are given by

$$f_{0,E}(l, \varphi) = \frac{1}{\sqrt{|l|}} [A \sin(\sqrt{2mEl}) + B \cos(\sqrt{2mEl})]. \quad (3.15)$$

For $E^2 + E'^2 \neq 0$ the coefficients A and B of the normalized state (3.15) satisfy the condition $|A|^2 + |B|^2 = \sqrt{2m}/2\pi^2$. Otherwise, if $E = E' = 0$ we have $|B| = \frac{1}{2\pi}$. Taking into account the identity

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad (3.16)$$

we find that in the limit $j \rightarrow 0$ of (3.11) only a part of the solution (3.15) referring to the rectilinear motion in the meridian is obtained. The lacking part could be derived as the limit $j \rightarrow 0$ of the function $J_{-\sqrt{\frac{1}{4} + \frac{j^2}{\sin^2 \alpha}}}(\sqrt{2mEl})$ by means of the identity

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cos z. \quad (3.17)$$

However, the Bessel function $J_{-\sqrt{\frac{1}{4} + \frac{j^2}{\sin^2 \alpha}}}(\sqrt{2mEl})$ has the index lesser than $-\frac{1}{2}$ and the “closure equation” (3.12) related to normalization cannot be applied. Such behavior of solutions (3.11) and (3.15) is a quantum scar of the instability of the classical rectilinear motion in the cone discussed in Sec. II.

4. Classical mechanics of the harmonic oscillator on a cone

We now discuss the motion of the classical particle on the double cone bound to its tip by harmonic oscillator potential. The corresponding Hamiltonian is (see (2.4))

$$H = \frac{p_l^2}{2m} + \frac{p_\varphi^2}{2ml^2 \sin^2 \alpha} + \frac{m\omega^2}{2} l^2, \quad (4.1)$$

and the Hamilton's equations are of the form

$$\begin{aligned} \dot{l} &= \frac{p_l}{m}, \\ \dot{\varphi} &= \frac{p_\varphi}{ml^2 \sin^2 \alpha}, \\ \dot{p}_l &= \frac{p_\varphi^2}{ml^3 \sin^2 \alpha} - m\omega^2 l, \\ \dot{p}_\varphi &= 0. \end{aligned} \tag{4.2}$$

Of course $J = p_\varphi = 0$ which leads to $\varphi = \text{const}$, refers to the case of the standard harmonic oscillator on a line (generator) with the solution

$$\begin{aligned} l &= l_0 \cos \omega t + \frac{p_{l0}}{m\omega} \sin \omega t, \\ p_l &= p_{l0} \cos \omega t - \omega m l_0 \sin \omega t. \end{aligned} \tag{4.3}$$

Now, using the expression on the energy such that

$$E = \frac{ml^2}{2} + \frac{J^2}{2ml^2 \sin^2 \alpha} + \frac{m\omega^2}{2} l^2, \tag{4.4}$$

we find for $J \neq 0$

$$l = \pm \sqrt{\frac{E + \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}} \sin 2\omega(t + C)}{m\omega^2}}, \quad J \neq 0, \tag{4.5}$$

where plus and minus sign correspond to the upper and lower nappe, respectively, and C is an integration constant. It thus appears that as with the case of the free motion, whenever $J \neq 0$ the particle remains in one concrete nappe during the time evolution. Therefore, the solution (4.3) describing the harmonic oscillations in a meridian around the point $l = 0$ turns out to be unstable (see Fig. 3). On the other hand, an immediate consequence of (4.5) is that for $J \neq 0$ the motion is finite. Namely, we find

$$\sqrt{\frac{E - \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}}}{m\omega^2}} \leq |l| \leq \sqrt{\frac{E + \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}}}{m\omega^2}}. \tag{4.6}$$

We remark that in the limit $J \rightarrow 0$ we obtain the correct amplitude $\sqrt{\frac{2E}{m\omega^2}}$ of the harmonic oscillations (4.3), however the period of harmonic oscillations

(4.3) with $J = 0$ is twice the period of oscillations (4.5) with $J \neq 0$. For the upper nappe the limit $J \rightarrow 0$ of (4.5) is

$$\lim_{J \rightarrow 0} l(t) = \sqrt{\frac{E}{m\omega^2}} \sqrt{1 + \sqrt{\frac{l_0^2 \omega^2 p_{l0}^2}{E^2}} \sin 2\omega t + \left(\frac{l_0^2 m \omega^2}{E} - 1\right) \cos 2\omega t}. \quad (4.7)$$

The trajectories on the cone (2.1) satisfying (4.2) involving the case of small J are shown in Fig 2.

Furthermore, using the second equation of (4.2) we get for $J \neq 0$

$$\varphi - \varphi_0 = \frac{\varepsilon(J)}{\sin \alpha} \left(\operatorname{arctg} \frac{E t \operatorname{tg} \omega(t + C) + \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}}}{\sqrt{\frac{J^2 \omega^2}{\sin^2 \alpha}}} - \operatorname{arctg} \frac{E t \operatorname{tg} \omega C + \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}}}{\sqrt{\frac{J^2 \omega^2}{\sin^2 \alpha}}} \right). \quad (4.8)$$

As with the case of the free motion we obtain in the limit $J \rightarrow 0$

$$\varphi - \varphi_0 = 0 \quad \text{or} \quad |\varphi - \varphi_0| = \pi. \quad (4.9)$$

Finally, using the first equation of (4.2) we derive the following formula on the momentum

$$p_l = \pm \frac{\sqrt{m} \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}} \cos 2\omega(t + C)}{\sqrt{E + \sqrt{E^2 - \frac{J^2 \omega^2}{\sin^2 \alpha}} \sin 2\omega(t + C)}}. \quad (4.10)$$

We point out that p_l is zero when the extremal values of $|l|$ are reached given by (4.6). In the limit $J \rightarrow 0$ this expression takes the form

$$p_l = \pm \sqrt{mE} \frac{\sqrt{\frac{l_0^2 \omega^2 p_{l0}^2}{E^2}} \cos 2\omega t - \left(\frac{l_0^2 m \omega^2}{E} - 1\right) \sin 2\omega t}{\sqrt{1 + \sqrt{\frac{l_0^2 \omega^2 p_{l0}^2}{E^2}} \sin 2\omega t + \left(\frac{l_0^2 m \omega^2}{E} - 1\right) \cos 2\omega t}}. \quad (4.11)$$

5. Quantum harmonic oscillator on a cone

Consider now a quantum particle moving on a double cone and bound to its tip by the harmonic oscillator potential. The corresponding Hamiltonian is

$$\hat{H} = \frac{\hat{p}_l^2}{2m} + \frac{\hat{J}^2}{2m\hat{l}^2 \sin^2 \alpha} + \frac{m\omega^2}{2} \hat{l}^2. \quad (5.1)$$

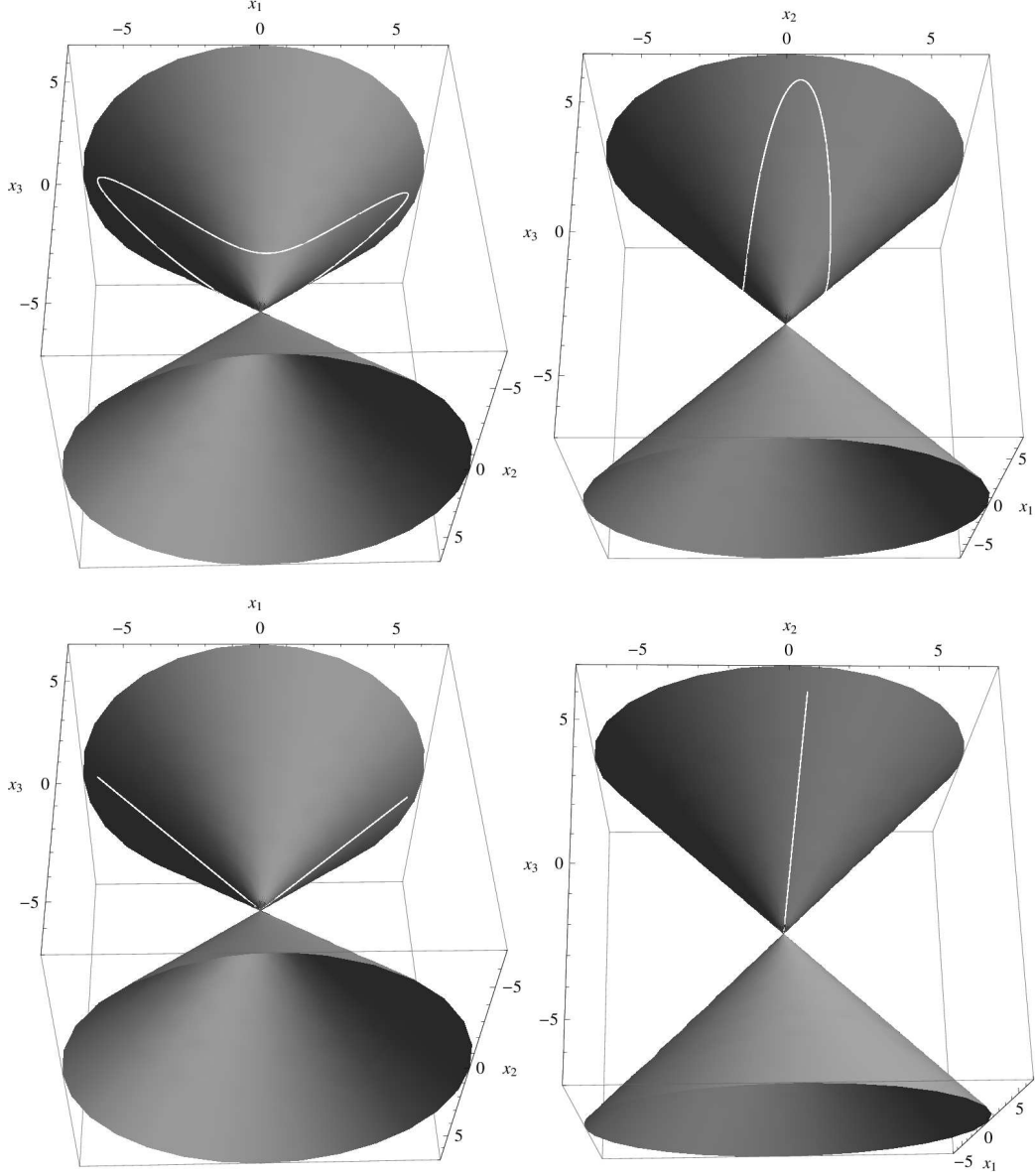


Figure 2: Top left: the trajectory of the harmonic oscillator on the cone given by (2.1) and (4.2) subject to the initial data $l_0 = 9$, $\varphi_0 = 0.1$, $p_{l0} = -1$, $J = 20$, and the parameters $\alpha = \pi/4$, $m = 1$ and $\omega = \sqrt{2}$. View from below. Top right: view of the right side of the trajectory on the cone presented in the figure on the left. Bottom left: the motion of the harmonic oscillator on the cone with the parameters and initial conditions the same as in the top figures besides of $J = 0.01$ referring to the limit (4.7) and (4.9). View from below. Bottom right: right-hand view of the bottom-left figure.

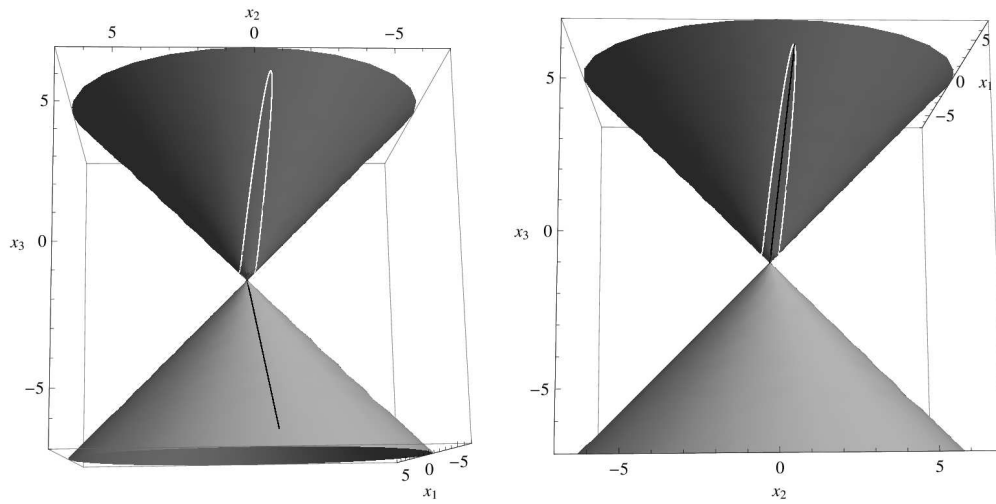


Figure 3: The illustration of the instability of the harmonic oscillations on the cone corresponding to the vanishing angular momentum. Left: white line refers to the solution to (4.2) with the same values of parameters and initial conditions as in the top figures in Fig. 2 besides of $J = 4$. Black line is the part of the segment from the lower nappe which is the trajectory of the harmonic oscillator with $J = 0$ (the remaining parameters and initial data coincide with those for the white trajectory). Right: the view of the figure on the left from the behind.

Consider the eigenvalue equation for the Hamiltonian

$$\hat{H}f_E(l, \varphi) = Ef_E(l, \varphi). \quad (5.2)$$

On separating variables and setting

$$f_{j,E}(l, \varphi) = \frac{1}{\sqrt{|l|}} e^{ij\varphi} \tilde{u}_{j,E}(l), \quad (5.3)$$

we can write (5.2) in the form

$$\frac{d^2 \tilde{u}_{j,E}(l)}{dl^2} + \left(2mE - \frac{j^2}{l^2 \sin^2 \alpha} - (m\omega)^2 l^2 \right) \tilde{u}_{j,E}(l) = 0. \quad (5.4)$$

We remark that the potential occuring in the Schrödinger equation (5.4) which is a sum of the harmonic oscillator and inverse square potential is referred to as the Smorodinsky-Winternitz potential [12], isotonic oscillator potential [13] or harmonic oscillator with “centripetal barrier” [14]. This potential is also closely related to the well-known Morse potential [15]. However, as far as we are aware, the Schrödinger equation (5.4) was not discussed in the literature in the context of the double cone. The quantum harmonic oscillator in the case of a cone with a single nappe or equivalently the plane with a deficit angle was studied in [4] and [6]. As mentioned earlier this case is much more complicated than that investigated herein dealing with quantum mechanics on a double cone, because one is forced to analyse self-adjoint extensions of symmetric operators representing physical observables [4, 5].

Now, making the ansatz

$$\tilde{u}_{j,E}(l) = C|l|^s e^{-\frac{m\omega l^2}{2}} w(m\omega l^2), \quad (5.5)$$

where C is the normalization constant, we find that whenever the parameter s satisfies

$$s(s-1) = \frac{j^2}{\sin^2 \alpha}, \quad (5.6)$$

which leads to

$$s = \frac{1}{2} \left(1 \pm \sqrt{1 + \frac{4j^2}{\sin^2 \alpha}} \right), \quad (5.7)$$

then w fulfils the Kummer equation

$$x \frac{d^2 w}{dx^2} + \left(s + \frac{1}{2} - x \right) \frac{dw}{dx} - \left(\frac{s}{2} + \frac{1}{4} - \frac{E}{2\omega} \right) w(x) = 0. \quad (5.8)$$

A numerically satisfactory solution to (5.8) with a good behavior at infinity is given by the Kummer function [16]

$$w(x) = U\left(\frac{s}{2} + \frac{1}{4} - \frac{E}{2\omega}, s + \frac{1}{2}, x\right). \quad (5.9)$$

Another notation for the Kummer function $U(a, b, z)$ is $\Psi(a, b, z)$ [17, 18]. Now, $U(a, b, x)$ is a polynomial when $a = -n$, where $n = 0, 1, 2, \dots$, and $U(a, b, x) \sim x^{-a}$ for $x \rightarrow \infty$. Demanding decreasing of $\tilde{u}_{j,E}(l)$ as $l \rightarrow \infty$ this leads to the quantization condition on energy

$$E_{j,n} = 2\omega \left(n + \frac{s}{2} + \frac{1}{4}\right). \quad (5.10)$$

Furthermore, the requirements $j \neq 0$ and convergence of the norm of the solution to (5.8) rule out the minus sign in (5.7), so

$$E_{j,n} = 2\omega \left(n + \frac{1}{2} + \frac{1}{4}\sqrt{1 + \frac{4j^2}{\sin^2 \alpha}}\right), \quad (5.11)$$

and

$$w(x) = U\left(-n, 1 + \frac{1}{2}\sqrt{1 + \frac{4j^2}{\sin^2 \alpha}}, x\right). \quad (5.12)$$

Using the identity [17]

$$U(-n, \alpha + 1, z) = (-1)^n n! L_n^\alpha(z), \quad (5.13)$$

where $L_n^\alpha(x)$ are the generalized Laguerre polynomials, we get

$$w(x) = (-1)^n n! L_n^{\frac{1}{2}\sqrt{1 + \frac{4j^2}{\sin^2 \alpha}}}(x). \quad (5.14)$$

Hence, we finally obtain the desired normalized (in the sense of $L^2(\mathbb{R}, dx)$) solution $\tilde{u}_{j,n}(l)$ to (5.4) such that

$$\begin{aligned} \tilde{u}_{j,n}(l) \equiv \tilde{u}_{j,E_n}(l) &= (-1)^n \sqrt{\frac{(m\omega)^{1+\frac{1}{2}\sqrt{1+\frac{4j^2}{\sin^2 \alpha}}} n!}{\Gamma\left(n+1+\frac{1}{2}\sqrt{1+\frac{4j^2}{\sin^2 \alpha}}\right)}} |l|^{\frac{1}{2}\left(1+\sqrt{1+\frac{4j^2}{\sin^2 \alpha}}\right)} e^{-\frac{m\omega l^2}{2}} \\ &\times L_n^{\frac{1}{2}\sqrt{1+\frac{4j^2}{\sin^2 \alpha}}}(m\omega l^2), \end{aligned} \quad (5.15)$$

and the normalized solution to the Schrödinger equation on the cone (5.2) of the form

$$f_{j,n}(l, \varphi) = \frac{1}{2\pi\sqrt{|l|}} e^{ij\varphi} \tilde{u}_{j,n}(l), \quad j \neq 0, \quad (5.16)$$

where the normalization is given by

$$\langle f_{j,n} | f_{j',n'} \rangle = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} dl |l| f_{j,n}^*(l, \varphi) f_{j',n'}(l, \varphi) = \delta_{jj'} \delta_{nn'}. \quad (5.17)$$

Up to the phase factor and normalization constant the solution (5.15) coincides with the solution to (5.4) introduced by Hakobyan *et al* in reference 12. However, the parameter k labeling the solution obtained in [12] had no physical interpretation. On the other hand, it follows immediately from (5.15) that if $k > \frac{1}{2}$, then this parameter can be related to the angular momentum j on the cone via $k = \frac{1}{2} \sqrt{1 + \frac{4j^2}{\sin^2 \alpha}}$.

Now, an immediate consequence of (5.3) and (5.4) is the following normalized solution to the eigenvalue equation (5.2) for $j = 0$:

$$f_{0,n}(l) = \frac{1}{2\pi\sqrt{|l|}} \left(\frac{m\omega}{\pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} e^{-\frac{m\omega l^2}{2}} H_n(\sqrt{m\omega} l), \quad (5.18)$$

referring to the energy

$$E_{0,n} = \omega(n + \tfrac{1}{2}), \quad (5.19)$$

where $H_n(z)$ are Hermite polynomials. Indeed, (5.4) for $j = 0$ is the Schrödinger equation for the standard harmonic oscillator on the line. Analogously as with the free motion, the solution (5.18) corresponding to $j = 0$ is different from the limit $j \rightarrow 0$ of the solution (5.16). In fact, taking into account the identity [19]

$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! x L_n^{\frac{1}{2}}(x^2), \quad (5.20)$$

we obtain the limit $j \rightarrow 0$ of $f_{j,n}(l)$ given by (5.16) such that

$$\lim_{j \rightarrow 0} f_{j,n}(l) = \frac{1}{2\pi\sqrt{|l|}} \left(\frac{m\omega}{\pi} \right)^{\frac{1}{4}} \frac{1}{\sqrt{2^{2n+1} (2n+1)!}} e^{-\frac{m\omega l^2}{2}} H_{2n+1}(\sqrt{m\omega} |l|). \quad (5.21)$$

The energy spectrum in the limit $j \rightarrow 0$ is

$$\lim_{j \rightarrow 0} E_{j,n} = \omega(2n + 1 + \tfrac{1}{2}). \quad (5.22)$$

We have thus obtained in the limit $j \rightarrow 0$ only a part of the solution corresponding to the case of the harmonic oscillator on a meridian (generator) related to the odd Hermite polynomials. As in the case of the free motion this observation is consistent with instability of the solution (4.3) of classical equations of motion describing harmonic oscillations around a point $l = 0$ in the meridian.

6. Conclusion

In this work we study the classical and quantum particle on the double cone in the case of the free motion and harmonic oscillator potential. An interesting peculiarity of the dynamics on the cone is the instability of the free motion in the generator on both classical and quantum level. This observation seems to be of importance for better understanding of quantum dynamics. Indeed, as with deterministic chaos the role of instabilities in quantum dynamics is far from clear. Referring to chaotic motion it is also worthwhile to point out that the unstable solution describing harmonic oscillations around a tip of the cone is rather exotic example of the unstable and bounded solution of a nonlinear dynamical system which does not show chaotic behavior. Furthermore, a remarkable property of the dynamics on the double cone investigated in this paper is the role played by the vertex which acts as a filter selecting from all possible motions the rectilinear one in the generator. Finally, we remark that the results of this paper provide a physical interpretation as a harmonic oscillator on the double cone for the one-dimensional Smorodinsky-Winternitz potential which can be connected with the Morse potential.

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